ISTEK BILGE KAGAN ANATOLIAN HIGH SCHOOL
Diploma Programme / PHYSICS

A Research Proposal on

Chaos and Double Pendulum : Measuring Change in Chaos

Research Question : How Lyapunov Exponents Change with Initial Angles in a Double Pendulum

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1. Introduction

The double pendulum consists of two pendulums attached together with the pivot of the second pendulum located at the end of the first. (Chen, 2008) While a simple pendulums can be easy to predict the next movement, double pendulums are more complex than simple pendulums. This complex structure depicts an enriching nonlinear dynamics and has captured the interest of mathematicians and physicians for decades.

1.1 Research Context

Chaotic systems can be termed systems that are sensitive to initial conditions. Often referred to as the " butterfly effect" (Lorenz, 1963) It was a milestone when Edward N. Lorenz introduced his nonlinear system, the "Lorenz System," and in a conference, he stated that a seagull's wing (which was later changed to a butterfly due to the shape of the attractor) could cause a hurricane on the other side of the world. What he actually meant was how the system is sensitive to the initial conditions. This feature of chaotic systems making impossible to make long-term predictions. Because of its easily observable chaotic behavior, the double pendulum is a perfect model for studying chaos. This is the reason why double pendulum selected in this study.

1.1.1 Research Questions

In this study, the research question is: 'How do Lyapunov exponents change with initial angles in double-pendulum systems?' This study will contain experiments that test how going to affect initial angles to Lyapunov exponents, which will be experimented with using Python and Python libraries. Lyapunov exponent can be termes as a simple way to measure "chaoticness"

1.2 Problem Statement

In this study, the estimated problem is the lack of a systematic, quantitative mapping between the initial angular conditions of a double pendulum and its resulting Lyapunov exponents.

2. Theory of Double Pendulum

2.0.1 System Definition and Variables

The double pendulum is a dynamical system consisting of two point masses, m_1 and m_2, connected by massless, rigid rods of lengths l_1 and l_2. The system is confined to motion in a vertical plane under uniform gravitational acceleration g.

The two generalized coordinates that completely define the system's configuration are:

- θ_1: The angle between the first rod and the vertical axis.
- θ_2: The angle between the second rod and the first rod.

The Cartesian positions of the masses are therefore given by:

$$x_1 = l_1 \sin\theta_1 \tag{2.1}$$

$$y_1 = -l_1 \cos\theta_1 \tag{2.2}$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \tag{2.3}$$

$$y_2 = -l_1 \cos\theta_1 - l_2 \cos(\theta_1 + \theta_2) \tag{2.4}$$

2.0.2 Derivation of Equations of Motion using Lagrangian Mechanics

$$\mathcal{L} = T - V \tag{2.5}$$

The equations of motion are then given by the Euler-Lagrange equations for each generalized coordinate q_i:

$$t\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad \text{for } i = 1, 2 \tag{2.6}$$

Kinetic Energy

The kinetic energy is the sum of the kinetic energies of both masses:

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \tag{2.7}$$

Differentiating the position equations (Eq. 2.1-2.4) with respect to time and substituting yields the kinetic energy in terms of $\dot{\theta}_1$ and $\dot{\theta}_2$:

$$\begin{aligned} T =&\frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2(\dot{\theta}_1^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ &+ m_2 l_1 l_2 \dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\cos\theta_2 \end{aligned} \tag{2.8}$$

Potential Energy

The potential energy, relative to the pivot point, is:

$$V = m_1 g y_1 + m_2 g y_2 \tag{2.9}$$

Substituting from equations 2.2 and 2.4:

$$V = -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos(\theta_1 + \theta_2) \tag{2.10}$$

The Lagrangian

The full Lagrangian is therefore:

$$\begin{aligned} \mathcal{L} =&\frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ &+ m_2 l_1 l_2 \dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\cos\theta_2 \\ &+ (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos(\theta_1 + \theta_2) \end{aligned} \tag{2.11}$$

Applying the Euler-Lagrange Equations

Applying the Euler-Lagrange equation for $q_i = \theta_1$ and $q_i = \theta_2$ leads to two coupled, second-order, non-linear differential equations. After simplification and rearrangement, they can be written in a matrix form suitable for numerical solving:

$$\begin{bmatrix} (m_1 + m_2)l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 \cos\theta_2 & m_2 l_2^2 + m_2 l_1 l_2 \cos\theta_2 \\ m_2 l_2^2 + m_2 l_1 l_2 \cos\theta_2 & m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{2.12}$$

Where the terms on the right-hand side contain the velocity and gravitational forces:

$$C_1 = -m_2 l_1 l_2 (2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2) \sin\theta_2 - (m_1 + m_2) g l_1 \sin\theta_1 - m_2 g l_2 \sin(\theta_1 + \theta_2)$$

$$C_2 = m_2 l_1 l_2 \dot{\theta}_1^2 \sin\theta_2 - m_2 g l_2 \sin(\theta_1 + \theta_2)$$

3. Preliminary Literature Review by ELICIT

According to the research question ”How do Lyapunov exponents change with initial angles in double-pendulum systems?”, Elicit searched across over 126 million academic papers from the Semantic Scholar corpus. Elicit retrieved the approximate 500 papers most relevant to the query. Report going to be shared with this text...

References

Chen, J. P. (2008). Chaos from simplicity : an introduction to the double pendulum.. Retrieved from `https://api.semanticscholar.org/CorpusID:151602808`

Lorenz, E. N. (1963). Deterministic nonperiodic flow. *Journal of the Atmospheric Sciences*, *20*, 130-141. Retrieved from `https://api.semanticscholar.org/CorpusID:15359559`