Euler Number: Applications in Diverse Disciplines

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Mathematics: Analysis and Approaches
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Abstract

In this study, one of the most popular constant numbers in mathematics, the Euler number, will be examined in different disciplines, from biology, physics, and astronomy, to data science, statistics, and financial analysis. Like the pi number, it is used commonly and helps us to understand progressive and growing concepts. Because of this feature of the Euler number, this constant is used in many disciplines. Furthermore, the Euler number has significant applications in combinatorics and number theory (Roa González et al.). Thus, it makes it one of the most essential constant numbers to understand the concepts of applied mathematics and theoretical mathematics.

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1. Introduction

Euler number is a constant number like pi, which was found by physicist and mathematician Leonhard Euler. It's a non-repeating number that never ends, beginning with 2.71828 (Kenton). Like pi, it is an irrational number. It is mostly used to calculate and measure the decay or growth of a particular factor over time, like compound interest. Moreover, Euler numbers are connected to special functions and have applications in the study of Bernoulli numbers and Stirling numbers (Roa González et al.). In contemporary disciplines, it contributes to diverse fields and subjects, beyond theoretical mathematics.

2. History of Euler Number

The mathematical constant Euler number has a rich history contributing to the foundation and advancement of various mathematical disciplines (Roa González et al.). The history of the Euler Number begins in 1683 with Jacob Bernoulli, when he was studying compound interest according to the University of St Andrews. In 1683, Jacob Bernoulli looked at the problem of compound interest and, in examining continuous compound interest, he tried to find the limit of

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

as n tends to infinity (O'Connor and Robertson). Jacob Bernoulli tried to use the binomial theorem to state that this number is between 2 and 3, and this was the birth of the e number. As far as is known in 2025, the number encountered in its own right is in 1690 with the letter of Leibniz to Huygens. According to the University of St Andrews, it is called b rather than e at that time. Furthermore, in 1697, Johann Bernoulli published *Principia calculi exponentialium seu percurrentium* to demonstrate that this number can be used in the calculation of various exponential series, and many results are achieved with term-by-term integration (O'Connor and Robertson). Most of the mathematical notations given by Euler like function for f(x) and most of people believe that, e number is letter e because of his name, which is a common misconception. It's believed that e refers to "*exponential*" but it may have just be the next vowel after "a" and Euler was already using the notation "a" in his work (O'Connor and Robertson) Euler's approximation for e to 18 decimal places,

$$e = 2.718281828459045235$$

without saying where this came from (O'Connor and Robertson). More calculations of decimal expansions came next. Boorman computed e to 346 decimal places in 1884 and discovered that his results matched Shanks's up to the 187th place, after which they diverged. Adams figured out the logarithm of e to the base 10 to 272 places in 1887.

3. Theory of Euler Number

3.0.1 Limit Definition

The most common definition, originating from Jacob Bernoulli's work on compound interest, is:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n (Wikipediacontributors) \tag{1}$$

More generally, for any real number x:

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n (Wikipediacontributors) \tag{2}$$

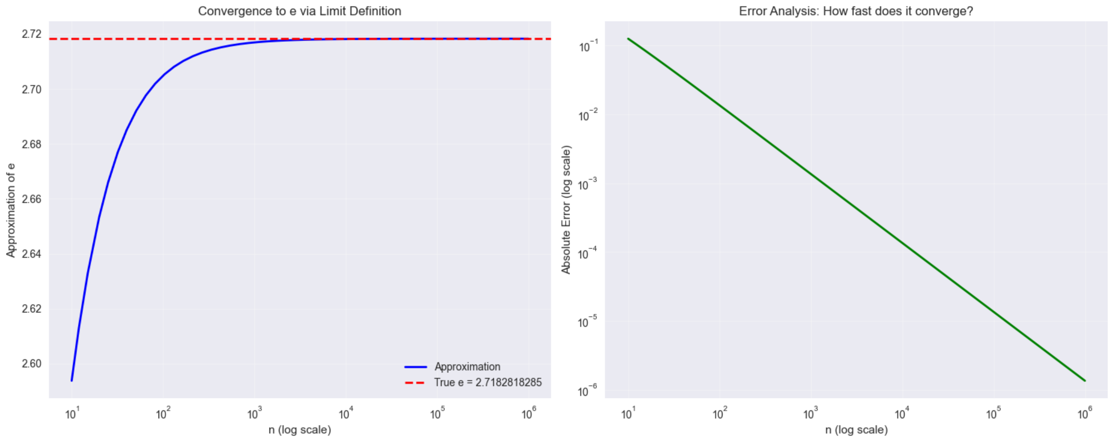


Figure 1: Convergence: $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$.

3.0.2 Series Definition

The exponential function can be defined as a power series, which provides one of the most useful computational definitions:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots (Wikipediacontributors) \tag{3}$$

Setting $x = 1$ gives the series representation for e itself:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots (Wikipediacontributors) \tag{4}$$

This series converges rapidly, making it excellent for numerical computation.

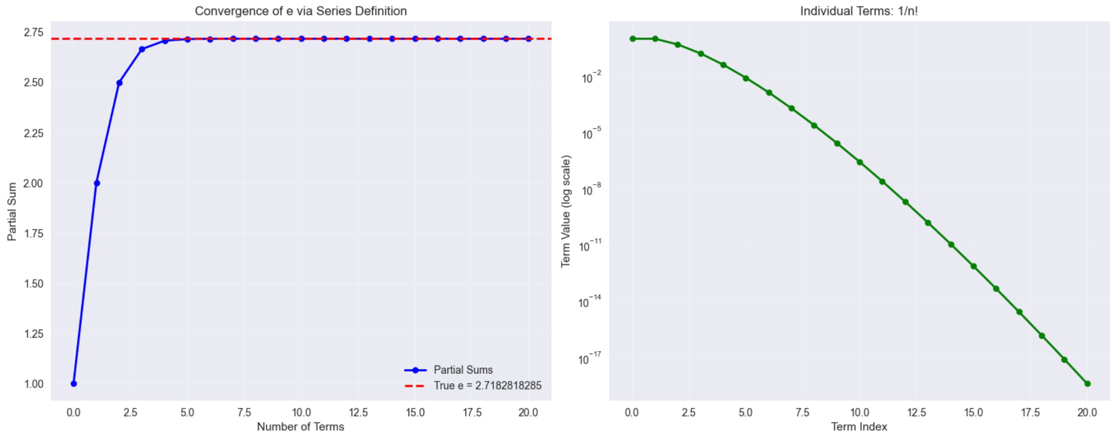


Figure 2: Convergence: $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

3.0.3 Integral Definition

The natural logarithm can be defined as the integral:

$$\ln x = \int_1^x \frac{1}{t}\,dt (Wikipediacontributors) \tag{5}$$

Since this function is one-to-one and onto R, its inverse exists and is defined as the exponential function:

$$e^x = \ln^{-1} x (Wikipediacontributors) \tag{6}$$

3.1 Fundamental Properties

3.1.1 Derivative Property

The exponential function $f(x) = e^x$ is unique in being equal to its own derivative:

$$\frac{d}{dx}e^x = e^x (Wikipediacontributors) \tag{7}$$

More generally, for any constant k:

$$\frac{d}{dx}e^{kx} = ke^{kx} (Wikipediacontributors) \tag{8}$$

This property makes e^x fundamental in solving differential equations.

3.1.2 Integral Property

Similarly, the integral of e^x is itself:

$$\int e^x\,dx = e^x + C (Wikipediacontributors) \tag{9}$$

3.2 Euler's Formula and Complex Numbers

One of the most remarkable relationships in mathematics is Euler's formula, which connects exponential and trigonometric functions:

$$e^{ix} = \cos x + i \sin x (Wikipedia contributors) \tag{10}$$

Setting $x = \pi$ yields Euler's identity:

$$e^{i\pi} + 1 = 0 (Wikipedia contributors) \tag{11}$$

This identity is celebrated for connecting five fundamental mathematical constants: 0, 1, e, i, and π.

3.3 The Natural Logarithm

The natural logarithm is the inverse function of the exponential function:

$$\ln x = \log_e x (Wikipedia contributors) \tag{12}$$

Its key properties include:

$\ln(xy) = \ln x + \ln y$

$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

$\ln(x^r) = r \ln x$

$\frac{d}{dx} \ln x = \frac{1}{x} (Wikipedia contributors)$

3.4 Numerical Value and Approximation

The numerical value of e begins as:

$$e \approx 2.71828\ 18284\ 59045\ 23536\ 02874\ 71352\ 66249\ 77572\ 47093\ 69995 \ldots \tag{13}$$

Using the series definition, we can obtain approximations:

e $\approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2.70833 \ldots$

$e \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = 2.71666 \ldots$

4. Applications of Euler Number

4.1 Modelling Population Growth

$$P(t) = P_0 \cdot e^{(b-d)t}(McKee) \tag{14}$$

Where P(t) is the population at time t, P0 is the initial population, b is the birth rate, d is the death rate, and (b-d) is the net growth rate. (McKee)

Example: A city with 10,000 people that grows by 2% each year will have:

$$P(10) = 10000 \cdot e^{(0.02)(10)} = 10000 \cdot e^{0.2} \approx 10000 \cdot 1.2214 = 12,214 people$$

after 10 years.

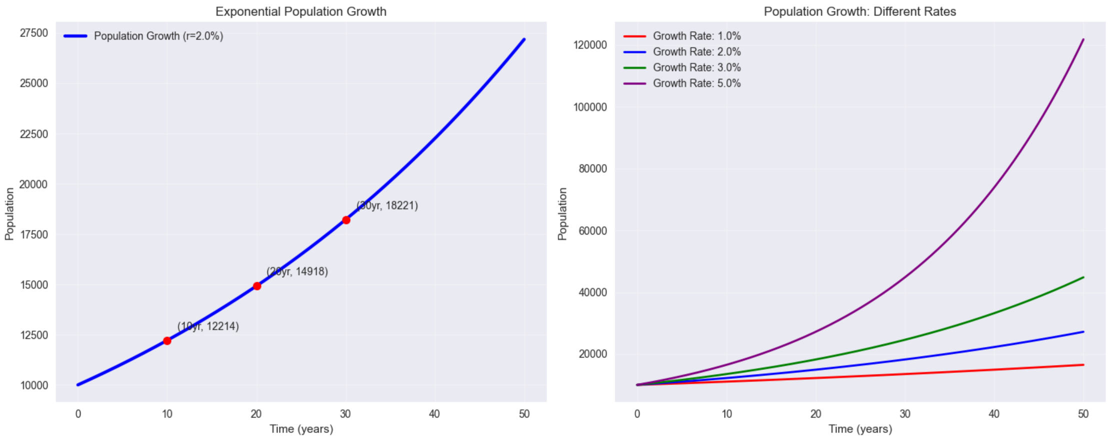


Figure 3: Exponential population growth

4.2 Modelling Bacterial Growth

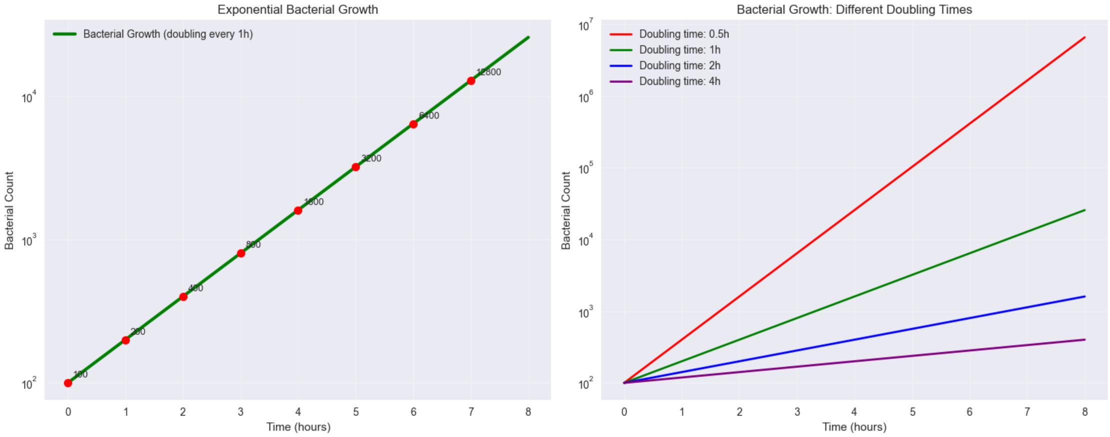
$$P(t) = P_0 \cdot e^{rt}(McKee) \tag{15}$$

Where P0 is the initial population, r is the growth rate, and t is time. (McKee)

Example: If bacteria double every hour (r = ln(2) 0.693), starting with 100 cells:

$$P(6) = 100 \cdot e^{(0.693)(6)} = 100 \cdot e^{4.158} \approx 100 \cdot 64 = 6,400 cells$$

after 6 hours.



Exponential Bacterial Growth

Bacterial Growth: Different Doubling Times

Figure 4: Bacterial growth modeling

4.3 Physics and Engineering

$$e^{ix} = \cos(x) + i\sin(x)(McKee) \tag{16}$$

It is used to represent sinusoidal waveforms in alternating current (AC) circuits. (McKee)

Example: For an AC voltage with amplitude 10V and frequency at time t = /2:

$$V = 10 \cdot e^{i(\omega t)} = 10 \cdot e^{i(\pi/2)} = 10 \cdot (\cos(\pi/2) + i\sin(\pi/2)) = 10 \cdot (0 + i \cdot 1) = 10i$$

This represents a purely imaginary voltage component at this instant.

4.4 Computer Science/Machine Learning and Cryptography

$$\sigma(x) = \frac{1}{1 + e^{-x}}(McKee) \tag{17}$$

This function maps inputs to outputs between 0 and 1, making it useful for classification tasks

(McKee).

Example: For an input feature x = 2 in a spam detection system:

$$\sigma(2) = \frac{1}{1+e^{-2}} = \frac{1}{1+0.1353} = \frac{1}{1.1353} \approx 0.88$$

This indicates an 88% probability that the email is spam.

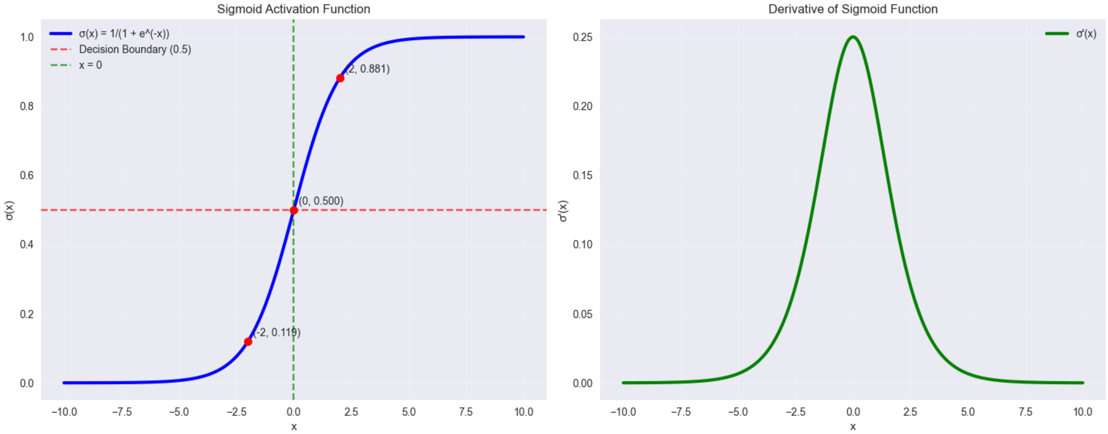


Figure 5: The sigmoid activation function $\sigma(x) = \frac{1}{1+e^{-x}}$

4.5 Geology/Radioactivity Decay

$$N(t) = N_0 \cdot e^{-\lambda t}(McKee) \tag{18}$$

Where N0 is the initial quantity of a radioactive isotope, is the decay constant, and t is time. (McKee)

Example: Carbon-14 has a half-life of 5730 years (= ln(2)/5730 0.000121). If a fossil has 25% of original carbon-14:

$$0.25 = e^{-0.000121t} \Rightarrow \ln(0.25) = -0.000121t \Rightarrow t = \frac{-1.3863}{-0.000121} \approx 11,457 years$$

The fossil is approximately 11,457 years old.

4.6 Finance and Compound Interest

$$A = P \cdot e^{rt} (McKee) \tag{19}$$

Where A is the final amount, P is the principal, r is the annual interest rate, and t is time in years.

Example: \$1,000 invested at 5% annual interest for 3 years with continuous compounding:

$$A = 1000 \cdot e^{(0.05)(3)} = 1000 \cdot e^{0.15} \approx 1000 \cdot 1.1618 = \$1,161.80$$

5. Reflection and Conclusion

This study examines Euler's number from different perspectives, including historical, theoretical, and applied. The various disciplines examined in this research have explored applications of Euler's number. While I was conducting this research, I learned that Euler's number is not simply a constant number but a number that can contribute diverse disciplines. This impressed me and showed me how a simple number can change paradigms at science engineering and more fields.

Works Cited

Kenton, Will. “Euler’s Number (e) Explained, and How It Is Used in Finance,” May 2025, Reviewed by David Kindness, www.investopedia.com/terms/e/eulers-constant.asp. Accessed 17 Oct. 2025.

McKee, Amberle. “Euler’s Number (e) Explained: Its Significance and Applications,” Apr. 2025, www.datacamp.com/tutorial/eulers-number. Accessed 17 Oct. 2025.

O’Connor, J. J., and E. F. Robertson. “The Number e.” *MacTutor History of Mathematics Archive*, Sept. 2001, mathshistory.st-andrews.ac.uk/HistTopics/e/. Accessed 17 Oct. 2025.

Roa González, Julián, et al. “An Overview of and Euler Numbers, Including Their History, Relevance, and Current and Future Applications.” *Axioms*, vol. 14, no. 3, 2025. https://doi.org/10.3390/axioms14030182.

Wikipedia contributors. “e (mathematical constant),” 2025, en.wikipedia.org/wiki/E_(mathematical_constant). Accessed 17 Oct. 2025.